

Scaling and universality in binary fragmenting with inhibition

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Abstract

We investigate a new model of binary fragmentation with inhibition, driven by the white noise. In a broad range of fragmentation probabilities, the power-law spatiotemporal correlations are found to arise due to self-organized criticality (SOC). We find in the SOC phase a non-trivial power spectrum of the temporal sequence of the fragmentation events. The $1/f$ behaviour is recovered in the irreversible, near-equilibrium part of this phase.

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The concept of self-organized criticality (SOC) [1] has attracted much attention as it has been demonstrated that it leads naturally to power-law distributions in space and time. The ubiquity of such correlations in nature and, in particular, $1/f$ noise in time [2], are expected to follow from the widespread occurrence of SOC in most dissipative many-body systems. In the fragmentation processes, the idea of universality comes from the observation that many experimental data on cluster-size (mass) distribution can be fitted by a single law $n_k \sim k^{-\tau}$, independently of interactions involved or orders of scale. Values of the exponent τ are known sufficiently accurately to exclude the possibility of grouping in few universality classes by the value of τ . There is here a striking resemblance with the fragmentation - inactivation binary (FIB) model[3, 4], which splits the set of all sequentially fragmenting systems into three classes. Two of them, at the transition line and in the shattering phase, exhibit power-law size-distribution, but the possible values of the exponent τ are smaller than 2 in one class (on the transition line) and are larger than 2 in the other one (the shattering phase) [5]. Despite this qualitative resemblance these two classes are of fundamentally different nature as manifested by the properties of the cluster-size fluctuations[4, 6]. It has been shown that the FIB model is able to reproduce quantitatively the multifragmentation results in nuclear heavy-ion collisions at intermediate energies[4].

In this Letter, we introduce the FIB automaton which is an alternative version of the FIB model, and show that it evolves into a SOC state with no characteristic time or length scales in a broad range fragmentation probabilities. We will show also that the occurrence of the anomalous power spectra $1/f^\beta$ ($1 \leq \beta < 2$) in

sequential fragmentation is associated with the SOC phase of the FIB automaton. In particular, $1/f$ spectrum is obtained in the limit of weak fragmentation activity.

The FIB automaton is defined on a one-dimensional lattice of linear size N ($i = 1, \dots, N$). The i th coordinate is an integer denoting a *mass* of the object. To each lattice site we assign an integer $n(i)$ which is to represent a *multiplicity of fragments* of mass i . The dynamical rules of the FIB automaton include the perturbation mechanism and the relaxation rules. Let us begin by the relaxation rules. One deals with clusters characterized by some conservative quantity, that we shall call the *cluster mass*. The cluster of mass N is relaxing via an ordered and irreversible sequence of steps. The first step is either a binary fragmentation occurring with the probability p_F or an inactivation, with probability $1 - p_F$. Once inactive, the cluster cannot be reactivated anymore. The successful fragmentation leads to two fragments, say $(N) \longrightarrow (j) + (N - j)$, with the mass partition probability $\alpha F_{j,N-j}$, where α is given by the normalization condition for coefficients $F_{i,j}$. The following step is either the fragmentation of the cluster (j) , with probability p_F , or its inactivation $(j) \longrightarrow (j)^*$, with probability $1 - p_F$. The relaxation process terminates when all clusters are inactive.

We consider the fragmentation process which is driven by the 'white noise' ($F_{j,l} = F_{l,j} = \text{const}$). It is assumed that the fragmentation probability p_F along the fragmentation cascade has a *fixed* value between 0 and 1, independent of the size of the fragmenting object, except for the monomers which are not allowed to break up [7]. Any event, either a fragmentation or an inactivation, occurs in the time interval between t and $t + dt$ after the cluster has been activated with a probability p_F .

For $p_F = \text{const}$, the FIB process can be mapped into the directed percolation on the Cayley tree which is a mean-field percolation. The fragmentation is a branching process, since the propagator of fragmentations is independent of mass. Each node is occupied with a probability 1 and at each occupied point at time l dt one chooses between three possibilities : fragmentation, inactivation, and 'no event' with respective probabilities p^2 , $(1 - p)^2$ and $2p(1 - p)$. The fragmentation probability is : $p_F = p^2/[p^2 + (1 - p)^2]$. This fragmentation is also analogous to the process of self-avoiding random walk, because the previously activated sites of this tree-like process 'repel' any subsequent reactivation. At each fragmentation, a given cluster is replaced by two descendants and the fragmentation multiplicity increases by one unit. The probability $P_N(\tilde{S})$ that a FIB branching process creates exactly M_0 fragments is for large N :

$$P_N[M_0 = \tilde{S}] = \frac{1}{2p_F} \frac{1}{2\tilde{S} - 1} \binom{2\tilde{S}}{\tilde{S}} [4p_F(1 - p_F)]^{\tilde{S}} \sim \tilde{S}^{-3/2} \exp(\alpha\tilde{S}) \quad \text{if } p_F \leq 1/2 \quad (1)$$

with $\alpha = \log[4p_F(1 - p_F)]$. For $p_F > 1/2$, the branching tree has an infinite size ($M_0 = \infty$). At last, $p_F = 1/2$ is the critical point of the FIB branching process as in this case the branching tree barely survives and a fragmentation multiplicity for large N is a power law with the exponent $3/2$. Moreover, the average value of the multiplicity, which is $\langle M_0 \rangle = (1 - p_F)/(1 - 2p_F)$ for $p_F < 1/2$ becomes infinite if $p_F > 1/2$. The value $p_F = 1/2$ is also a critical point of the avalanche process for the undirected sandpile process in the mean-field limit[8]. Notice however, that the sequential fragmentation is also critical from the point of view of the shattering

transition for any value of p_F , even though the branching tree size-distribution is modified.

The FIB automaton is perturbed by adding one mass unit to the randomly chosen fragment (j) on the lattice $\{n(i), i = 1, \dots, N\}$. The cluster of mass N is unstable and, therefore, the fragmentation avalanche is started if the perturbation produces a cluster N . Once we have chosen the perturbation mechanism and the relaxation algorithm then the algorithm of the temporal evolution of the FIB automaton goes as follows :

- (1) Specify an initial configuration $\{n(i), i = 1, \dots, N\}$.
- (2) Whenever $n(N) \neq 0$, the cluster of size N breaks into smaller fragments according to the relaxation algorithm of FIB model until all branches of the FIB cascade become inactive.
- (3) The fragment multiplicity at all sites is updated, increasing by the amount of produced inactive clusters.
- (4) Choose a fragment (j) at random and increase its mass by one unit: $j \rightarrow j + 1$. Change the fragment multiplicities at sites j : $n(j) \rightarrow n(j) - 1$ and $j + 1$: $n(j + 1) \rightarrow n(j + 1) + 1$. Return to step 2 .

For $p_F < 1/2$, after a transient period, a stationary state of FIB automaton is reached in which the normalized average size of the cluster $\sigma(t) = (1/N) \sum_j j n(j)$ exhibits small fluctuations around its asymptotic value :

$$\bar{\sigma} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \sigma(t) = \frac{1 - 2p_F}{2 - 2p_F} . \quad (2)$$

This is the 'high-viscosity' regime, where the accumulation mechanism of the FIB

automaton compensates exactly the relaxation mechanism and the system drives alone into a steady state with parameters independent of the lattice size. We shall see below that the steady state of the FIB automaton in this regime has all features of SOC state. In the upper part of Fig. 1 we show for $p_F = 0.1$ the dependence of the average size of the clusters on time, where a unit time-step is one update of the whole lattice $\{n(i), i = 1, \dots, N\}$. For $p_F \geq 1/2$, the asymptotic average cluster-size $\bar{\sigma}$ depends explicitly on the lattice size and equals $\sim (\ln N)^{-1}$ for $p_F = 1/2$ and $\sim 1/[(2 - 2p_F)\zeta(2p_F)]N^{1-2p_F}$ for $p_F > 1/2$. Thus, $\bar{\sigma}$ becomes equal zero in the thermodynamic limit $N \rightarrow \infty$. This is the 'low-viscosity' regime of the FIB relaxation mechanism, where the cluster-mass accumulation is completely destroyed. We have found, that for any p_F , the cluster-size distribution in the steady state of the FIB automaton exhibits a power-law behaviour. However, in contrast to the original FIB model [3, 4], it does not manifest any finite-size effects even in small lattices (see the lower part of Fig. 1). The average slope of the cluster-size distribution in the steady state exhibits small fluctuations about its asymptotic value $\bar{\tau} = 2p_F$.

Before we discuss the statistical properties of FIB automata, let us first define the *instantaneous dissipation rate* for the avalanche e :

$$f_{(e)}(t) = \sum_k \chi_k^{(e)}(t) \quad (3)$$

$f_{(e)}(t)$ is an indicator function of unstable clusters at a time t in the fragmentation avalanche. $\chi_k^{(e)}(t)$ in (3) is the characteristic function of the cluster k and equals either 1, for $t \in [t_k, t'_k]$ where t_k is the instant of formation of the cluster k and

t'_k the time when it disappears, or 0 otherwise. The summation goes over all clusters in the avalanche e . At a given time t , the value of the characteristic function $f(t)$ is then just the number of *active* clusters.

The lifetime distribution of fragmentation avalanches are shown in Fig. 2 for few selected values of p_F . One may notice a power-law behaviour $P[T = t] \sim t^{1-\alpha'}$ for $p_F < 1/2$, i.e. in the 'high-viscosity' regime. The values of the exponent α' for different values of p_F are summarized in Table 1.

The avalanche-size is defined as the *total dissipation* of the fragmentation avalanche :

$$\hat{s} = \int_0^\infty f(t)dt = \sum_k (t'_k - t_k) \sim \tilde{M}_0 \tilde{T} \quad , \quad (4)$$

This is just the sum of the lifetimes of all the clusters which have appeared in the sequence of breaks, where \tilde{M}_0 and \tilde{T} are the average multiplicity and the total fragmentation time for a given event respectively. For $p_F < 1/2$, the value of \tilde{M}_0 is a function of p_F but does not depend on the size of the fragmenting object. Hence, we expect that $\hat{s} \sim t^{\gamma_1}$ with $\gamma_1 = 1$. This is verified by the calculated size-distribution, which exhibits also a power-law behaviour $P[\hat{S} = \hat{s}] \sim \hat{s}^{1-\tau'}$ with $\tau' = \alpha'$. For $p_F > 1/2$, i.e. in the 'low-viscosity' regime, the branching tree of the fragmentation process survives until the low mass cutoff for monomers. In this case, the distribution of both avalanche-lifetimes and avalanche-sizes loose their scale-invariant features and are dominated by the cutoff scale. Consequently, only for relatively small time and size scales, the distributions $P[T = t]$ and $P[\hat{S} = \hat{s}]$ are consistent with a power-law behaviour. In this range of scales, $\alpha' = \tau' = 2$, independent of p_F .

The sequential fragmentation in FIB automaton has properties of the SOC state for all p_F in the 'high-viscosity' regime $0 < p_F < 1/2$. For each p_F , the stationary state of the FIB automaton has different features as manifested *e.g.* by different asymptotic values of the average cluster-size $\bar{\sigma}$ or the average slope of the cluster-size distribution $\bar{\tau}$. For each value of the fragmentation probability p_F ($< 1/2$), FIB model provides a different mean-field realization of a SOC state with its particular exponents of the power-law spatiotemporal distributions. In this sense, it is appropriate to speak about a SOC phase of FIB automaton for $0 < p_F < 1/2$. The extreme points: $p_F = 0$ and $1/2$ at the two ends of the SOC phase have a particular significance as they correspond to the stationary limit of the fragmentation process and to the critical point of the branching process respectively. Below, we shall show, that they play also a special role in characterizing the anomalous power spectra $S(f) \sim 1/f^\beta$ with $1 \leq \beta \leq 2$.

One of the main aims behind SOC studies was the search for a generic explanation of $1/f$ flicker noise[2] whose widespread occurrence is one of the great unresolved problems in physics. The concept of SOC state is an attempt to find a general principle behind ubiquity of $1/f$ noise in apparently unrelated physical systems. As pointed out before[9, 10, 11], the spatiotemporal scaling in the SOC state does not necessarily manifest itself in the nontrivial features of the noise spectrum. Indeed, the power spectrum of the cellular automaton model of Bak, Tang and Wiesenfeld[1] is $1/f^2$ as for the random walk.

Power spectrum of linearly superimposed fragmentation avalanches can be cal-

culated directly. For that let us define the *total dissipation rate* :

$$j(t) = \sum_e f_{(e)}(t - \tau_{(e)}) \quad (5)$$

where $\tau_{(e)}$ is a random instant of time when a fragmentation in the event e has started and the summation goes over all events generated by the FIB automaton.

The power spectrum of $j(t)$:

$$S(f) = \left\langle \left(\sum_k \frac{\cos ft'_k - \cos ft_k}{f} \right)^2 + \left(\sum_k \frac{\sin ft'_k - \sin ft_k}{f} \right)^2 \right\rangle \quad (6)$$

is shown in Fig. 3 for few values of p_F . The function $S(f)$ indeed shows a power law behaviour $\sim 1/f^\beta$ even in relatively small systems ($N = 2048$). The values of the power law exponent β for different values of p_F are shown in Table 1. For $p_F \geq 1/2$, the power spectrum is those of a random walk, *i.e.* $1/f^2$. For $p_F < 1/2$, *i.e.* in the SOC phase of the FIB automaton, the power spectrum is anomalous ($1 < \beta < 2$) in the frequency range $f_0 < f < f_1$. For $f < f_0$, one finds a 'white noise' power spectrum and the low-frequency limit f_0 is independent of N . For $f > f_1$, the power spectrum becomes $1/f^2$ and the high-frequency limit f_1 grows linearly with N . High frequencies correspond to short fragmentation times, where one is sensible to the fragmentation of the first cluster of size N . Since this process is unique, one expects $1/f^2$ spectrum as soon as $f \gg 1/\tau_N$, where $\tau_N \sim p_F/(N-1)$ is a typical time of disappearing of the first cluster in the fragmentation avalanche. The exponent β becomes 1 in the limit $p_F \rightarrow 0$, *i.e.* when approaching the stationary limit of the FIB automaton. The exponent β of the power spectrum $S(f)$ is related to the exponents α', τ' of the lifetime- and size- distributions by the relation :

$$\beta = (4 - \tau') \frac{2 - \alpha'}{2 - \tau'} \quad (7)$$

This relation has been derived by Christensen et al. (see eqs. (36), (39) and (40) of ref.[10]) for sandpile cellular automata. Numerical verification of this relation is shown in Table 1. The validity of (7) for $p_F \geq 1/2$ is less certain as the power-law behavior in distributions $P[T = t]$ and $P[\hat{S} = \hat{s}]$ is here restricted to relatively short times only.

The proposed mechanism for anomalous activity spectrum due to critical fragmentation is a ‘bulk’ phenomenon associated with the ‘mass’ fragmentation. It is an irreversible phenomenon, associated with the dynamical SOC phase of the fragmentation process. The coherence in the activity spectrum over all time scales ($\beta \rightarrow 1$) is found in the limit of dying out fragmentation activity ($p_F \rightarrow 0$), *i.e.* close to the stable equilibrium of the system without external driving. The ubiquity of the $1/f$ phenomena follows once it has been established that the near-equilibrium dynamics of most dissipative many-body systems is determined by the SOC state which is an attractor to the complicated many-body dynamics. In contrast to the explanation of the flicker noise, based on the deterministic spring - block models[12] or the lattice-gas model[13] the present model has also an inherent local conservation law, the cluster ‘mass’ in each fragmentation vertex.

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Figure captions

Fig. 1

In the upper part, the normalized average size of the cluster σ is plotted as a function of time for the FIB automaton with $N = 512$ and $p_F = 0.1$. The solid line shows the asymptotic value $\bar{\sigma} = 4/9$ when $t \rightarrow \infty$. The bottom part of the figure shows the cluster-size distribution of the FIB automaton in the steady state for $p_F = 0.1$ (filled circles) and 0.25 (filled triangles) in the SOC phase, for $p_F = 0.5$ (open circles) at the critical point of the branching process and for $p_F = 0.75$ (open squares) above it. The slope of the power-law distributions $n(k)$ is $\tau = 2p_F$, in accordance with results found earlier in the FIB model[4]. Notice a total absence of the finite-size effects even for $k = N$. The data correspond to 10^6 fragmentation avalanches with $N = 256$.

Fig. 2

Lifetime distributions of the fragmentation avalanches for different values of p_F . Significant deviations from the power-law distribution for large times are seen for $p_F \geq 0.5$ and are due to the appearance of 'infinite' branching trees which reach the cutoff for monomers. The data correspond to 10^6 fragmentation avalanches with $N = 2048$. The same signatures are used as in the bottom part of Fig. 1.

Fig. 3

Power spectra corresponding to the lifetime distributions shown in Fig. 2. The same

signatures are used as in the bottom part of Fig. 1.

Table 1

Exponents for different values of the fragmentation probability p_F . In the last column, the sum of calculated exponents β and τ' is given. $\beta + \tau' = 4$ for sandpile cellular automata as verified by Christensen et al[10].

| p_F | β | $\alpha' = \tau'$ | $\beta + \tau'$ |
|-------|-----------------|-------------------|-----------------|
| 0.05 | 1.0 ± 0.1 | 3.0 ± 0.3 | 4.0 ± 0.4 |
| 0.10 | 1.16 ± 0.05 | 2.81 ± 0.05 | 3.97 ± 0.1 |
| 0.25 | 1.50 ± 0.03 | 2.53 ± 0.10 | 4.03 ± 0.1 |
| 0.40 | 1.91 ± 0.05 | 1.90 ± 0.10 | 3.81 ± 0.1 |
| 0.50 | 2.00 ± 0.02 | — | — |
| 0.75 | 2.00 ± 0.05 | — | — |